Rotations and Angular Momenta

Finite Versus Infinitesimal Rotations

Consider a vector

$$\mathbf{V} = \begin{pmatrix} V_x & V_y & V_z \end{pmatrix}^{\mathsf{T}} ,$$

after a rotation

$$\begin{pmatrix} V_x' \\ V_y' \\ V_z' \end{pmatrix} = R \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

with

$$R^{\mathsf{T}} R = R R^{\mathsf{T}} = 1 ,$$

leading to a property

$$\mathbf{V}^{\mathsf{T}}\mathbf{V} = \mathbf{V}^{\mathsf{T}} R^{\mathsf{T}} R \mathbf{V} ,$$

$$V_x'^2 + V_y'^2 + V_z'^2 = V_x^2 + V_y^2 + V_z^2 .$$

Define a rotation operator about the z-axis by angle ϕ,

$$R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We are particularly interested in an infinitesimal form of R_z:

$$R_z(\epsilon) = \begin{pmatrix} 1-\frac{\epsilon^2}{2} & -\epsilon & 0 \\ \epsilon & 1-\frac{\epsilon^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad \epsilon \to 0 .$$

Likewise, we have

$$R_x(\epsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{\epsilon^2}{2} & -\epsilon \\ 0 & \epsilon & 1-\frac{\epsilon^2}{2} \end{pmatrix} ,$$

and

$$R_y(\epsilon) = \begin{pmatrix} 1-\frac{\epsilon^2}{2} & 0 & \epsilon \\ 0 & 1 & 0 \\ -\epsilon & 0 & 1-\frac{\epsilon^2}{2} \end{pmatrix} .$$

Elementary matrix manipulations lead to

$$R_x R_y = \begin{pmatrix} 1 - \frac{\epsilon^2}{2} & 0 & \epsilon \\ \epsilon^2 & 1 - \frac{\epsilon^2}{2} & -\epsilon \\ -\epsilon & \epsilon & 1 - \epsilon^2 \end{pmatrix}$$

$$R_y R_x = \begin{pmatrix} 1 - \frac{\epsilon^2}{2} & \epsilon^2 & \epsilon \\ 0 & 1 - \frac{\epsilon^2}{2} & -\epsilon \\ -\epsilon & \epsilon & 1 - \epsilon^2 \end{pmatrix}$$

$$R_x R_y - R_y R_x = \begin{pmatrix} 0 & -\epsilon^2 & 0 \\ \epsilon^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = R_z(\epsilon^2) - 1 \ ,$$

where all terms of order higher than ϵ^2 have been ignored.

Infinitesimal Rotations in Quantum Mechanics

Given a rotation operation characterized by a orthogonal 3×3 matrix R, associate an operator $\mathcal{D}(R)$ in the appropriate ket space such that

$$|\alpha\rangle_R = \mathcal{D}(R)|\alpha\rangle \ .$$

- For describing a spin$-1/2$, system with no other degrees of freedom, $\mathcal{D}(R)$ is a 2×2 matrix;
- for a spin-1 system, $\mathcal{D}(R)$ is a 3×3 matrix.

The appropriate infinitesimal operators could be written as

$$\hat{U}(\epsilon) = 1 - \mathrm{i}\hat{G}\epsilon \ , \quad \hat{G} : \ \text{Hermitian}$$

We therefore define the angular-momentum operator $\hat{J}_k$ for an infinitesimal rotation around the kth axis by angle $\mathrm{d}\phi$ can be obtained by letting

$$\hat{G} \to \frac{\hat{J}_k}{\hbar} \ , \quad \epsilon \to \mathrm{d}\phi$$